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Maximal rank for planar singularities of multiplicity 2

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Abstract

We prove that general unions of singularity schemes of multiplicity two in the projective plane have maximal rank.

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1. Introduction

It has been known for a long time that given a set of $r \neq 2, 5$ points in general position in the plane and a positive integer d , the linear system of all curves of degree d singular at the r points has dimension $\max\{-1, d(d+3)/2 - 3r\}$. In other words, each singularity imposes 3 linearly independent conditions or there are no curves with the required singularities, except in the aforementioned cases. Early references to this result can be traced back to Palatini [21]. The standard modern reference is Hirschowitz [18].

General curves in the linear systems just described have ordinary nodes as their only singularities; it is natural to ask about linear systems of curves with more complicated singular points, and under which hypotheses the conditions imposed by the singularities are linearly independent. By general principles it is clear that they will be independent if the degree is large enough, and there are recent results due to Shustin and his collaborators that

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provide bounds for what “large enough” must mean if the “position” of the singularities is general. These bounds of [15] and [24] are valid for any type of singularity, but they are not sharp, and for some kinds of singularities (such as nodes, as above) it is possible to do better.

D. Barkats proved in [5] that the linear system of all curves of a given degree with ν ordinary nodes and κ ordinary cusps at given (general) points and with given (general) tangents for the cusps has dimension $\max\{-1, d(d+3)/2 - 3\nu - 5\kappa\}$, except in the two cases already encountered (2 or 5 nodes) and when there are two cusps. In other words, also in this case each singularity imposes linearly independent conditions or there are no curves with the required singularities. The same is true for node and tacnode singularities [22], except when the orders of the nodes and tacnodes add up to 2 or 5 (they are coalescences of the classical 2- and 5-node cases). In Arnold’s notation, this means that a collection of singularities of types A_1 and A_2 imposes independent conditions, as does a collection of singularities of types A_k where every k is odd. In this work (Theorem 5.3) we prove that this holds in fact for every collection of singularities of multiplicity two (i.e., of types A_k with k arbitrary) with the only exceptions already known.

In order to precisely state what “imposing a singularity of a given type in a given position” and “general position” mean we need some algebro-geometric language. Let us fix the setting first. Except if otherwise stated, we work over an algebraically closed field k of arbitrary characteristic. A type of singularity means an equivalence class of germs of plane curve under equisingularity—two singular points are equisingular if their embedded resolutions have the same combinatorics. If $k = \mathbb{C}$ is the complex field, then equisingularity is the same as topological equivalence (in a neighborhood of the singular point). We work in the projective plane $\mathbb{P}^2 = \mathbb{P}_k^2$ although, as we deal with points in general position, it is more or less indifferent to use an affine or projective setting.

The embedded resolution of a singular point of a plane curve consists in blowing up the point and all singular points of its successive strict transforms, so that at the end of the (finite) process one gets a surface in which the strict transform of the curve is nonsingular and its total transform is a normal crossings divisor. To the *cluster* of points that have been blown up we associate a combinatoric invariant, the weighted *Enriques diagram*, which is a tree whose vertices represent the points, whose edges represent their proximity relations and comes with integral weights, that represent their multiplicities—a point is proximate to another if it lies in (the strict transform of) its exceptional divisor. Two singular points are equisingular if and only if the Enriques diagrams of the associated clusters coincide. We usually denote clusters by capital letters as K , whereas the weights are denoted by \mathbf{m} , an Enriques diagram is D and a weighted diagram is (D, \mathbf{m}) .

The reader may find the basics on clusters and Enriques diagrams in [7]. Here we need mainly the *complete ideals* defined by weighted clusters; let us briefly recall some of the basic facts concerning them. Let O be a point in the plane where the curve C has a singularity. Let \mathcal{O} be the (two-dimensional, regular) local ring at O (so the germ of C is defined as $f = 0$ for some $f \in \mathcal{O}$), and let K be the cluster of the embedded resolution of C . Then the set $I_{(K, \mathbf{m})}$ of all $g \in \mathcal{O}$ such that the germ of curve $g = 0$ goes through the points of K with (virtual) multiplicities at least as big as those of C is an ideal, which is \mathbf{m} -primary (\mathbf{m} being the maximal ideal of \mathcal{O}) and complete. For general $g \in I_{(K, \mathbf{m})}$, $g = 0$ is equisingular to C . The equisingularity type is determined by the class of g modulo \mathbf{m}^n for some n ,

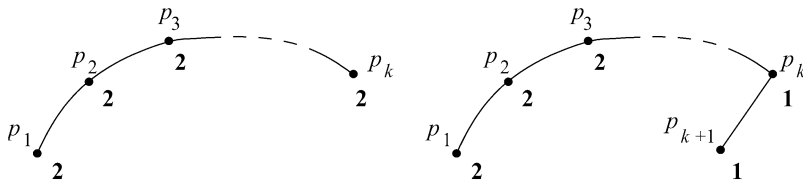


Fig. 1. Enriques diagrams of a tacnode (left, A_{2k-1}) and a cusp (right, A_{2k-2}). The multiplicities (weights) are in boldface. In both cases every point is proximate to its predecessor, but the last point p_{k+1} of a cusp is proximate to p_{k-1} as well, so it is a *satellite*; this is represented in the diagram by a straight segment. These are all the singularities of multiplicity 2.

and $I_{(K, \mathbf{m})}$ is \mathfrak{m} -primary, so one may use \mathcal{O} or its completion with respect to the maximal ideal, if it simplifies matters.

If C is a curve with several singular points, we associate to it a cluster that is the disjoint union of the clusters of all its singularities, and the corresponding Enriques diagram (which is now a forest rather than a tree). The set $Cl(D)$ of all clusters with the same Enriques diagram D has a natural structure of quasiprojective algebraic variety [23]; whenever we state some claim about singularities of type (D, \mathbf{m}) in general position we mean that the claim holds for singularities whose cluster lies in a Zariski open set of $Cl(D)$. Figure 1 shows the Enriques diagrams that appear for singularities of multiplicity 2.

Let K be a cluster of points in \mathbb{P}^2 with some weights that correspond to a singularity type. Let $I_{(K, \mathbf{m})}$ denote the ideal sheaf supported at the proper points of K which is locally defined as above, by the condition of going through the points of K with the assigned multiplicities, and let $Z_{(K, \mathbf{m})}$ denote the (zero-dimensional) subscheme of \mathbb{P}^2 defined by $I_{(K, \mathbf{m})}$. For every positive $d \in \mathbb{Z}$, the twisted global sections $\Gamma(I_{(K, \mathbf{m})}(d))$ are homogeneous polynomials of degree d defining curves that go through the points of K with the assigned multiplicities, and $\mathbb{P}(\Gamma(I_{(K, \mathbf{m})}(d)))$ is the linear system of curves of degree d with the assigned singularities at the assigned positions. If d is high enough, then general curves in $\mathbb{P}(\Gamma(I_{(K, \mathbf{m})}(d)))$ do indeed have the singularity type given by the Enriques diagram of K . The conditions imposed by K are independent or there are no curves of degree d containing $Z_{(K, \mathbf{m})}$ if and only if the canonical map

$$k[x, y, z]_d \cong \Gamma(\mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow \Gamma(\mathcal{O}_{Z_{(K, \mathbf{m})}}(d)) \cong \Gamma((\mathcal{O}_{\mathbb{P}^2}/I_{(K, \mathbf{m})})(d))$$

is either surjective or injective, i.e., has maximal rank. If this happens for a given (K, \mathbf{m}) and for all d we say that $Z_{(K, \mathbf{m})}$ has maximal rank. Thus for instance the first result mentioned in the introduction may be rephrased in more fancy words by saying that, if D is the diagram consisting of r unconnected vertices—no edges—, K is general in $Cl(D)$, and we take all weights equal to 2 then $Z_{(K, \mathbf{m})}$ has maximal rank, and we claim that the same is true if (D, \mathbf{m}) is a union of weighted diagrams of the types shown in Fig. 1.

It may be good to warn that the schemes $Z_{(K, \mathbf{m})}$ obtained with a fixed weighted Enriques diagram (D, \mathbf{m}) need not be isomorphic in general. The class of schemes isomorphic to a given $Z_{(K, \mathbf{m})}$ is contained in the class of all schemes $Z_{(K, \mathbf{m})}$, where K has diagram (D, \mathbf{m}) ; for almost all (D, \mathbf{m}) there is a nontrivial moduli space of such schemes. However, for types (D, \mathbf{m}) of multiplicity two all $Z_{(K, \mathbf{m})}$ are indeed isomorphic (because

A_k -singularities have no moduli, see [4]) so in the sequel we seldom mention isomorphism classes of zero-dimensional schemes.

To show that general schemes of a given class have maximal rank it is often useful to use *specialization* and *semicontinuity*: if one has a flat family of schemes Z_t parameterized by some smooth scheme $t \in T$, such that for some special value of the parameter $t = 0$ the scheme Z_0 has maximal rank, then the principle of semicontinuity [17, Chapter III, 12] tells us that general members of the family have maximal rank. The strategy of our proof consists in a sequence of specializations which furnish a family Z_t whose general members are of type $Z_{(K, \mathbf{m})}$ (where the Enriques diagram of K is a union of diagrams of the types shown in Fig. 1) and where Z_0 is known to have maximal rank.

Let (D, \mathbf{m}) be a given union of weighted diagrams of the types shown in Fig. 1. The first specialization simplifies matters by reducing to a family of schemes supported at a single point. For every cluster K with diagram D there is a smooth curve C going through all the free points of K (i.e., through the subcluster K' consisting of every point of the tacnodes and every point of the cusps except the last one, weighted with multiplicity 1 at all the points). It is enough to pick C of high enough degree and consider the complete (curvilinear) ideal associated to K' with these weights. We allow the base points to move on C , and specialize them to “collide” at a single point, giving as a flat limit a zero-dimensional scheme supported at a single point and contained in $2C$, the one-dimensional scheme whose equation is the square of that of C .

The specialization used leads to schemes that are not singularity schemes, because they are not defined by complete ideals; their defining property is to be contained in the double of a smooth curve, and we call such schemes 2-curvilinear schemes (see Section 4). Actually we prove a maximal rank statement for 2-curvilinear schemes; as far as we know, this is the first place where a maximal rank result is proved that involves schemes whose defining ideals are not complete. This can be understood as a generalization of the well-known fact that general curvilinear schemes have maximal rank (see [8] in arbitrary dimension, or apply [6] in dimension 2).

There are two numerical invariants naturally associated to a 2-curvilinear scheme Z . The first is the length N of Z , and the second is the maximal contact ℓ of Z with a smooth curve whose double contains Z ; they satisfy the inequalities $0 \leq \ell \leq N \leq 2\ell$. Our main result is the following.

Theorem 1.1. *Let N, ℓ be two positive integers with $0 \leq \ell \leq N \leq 2\ell - 1 - 3\sqrt{N - \ell}$. Then for every isomorphism class of 2-curvilinear schemes Z whose length is N and whose maximal contact with smooth curves whose double contains Z is ℓ , general members of the class in \mathbb{P}^2 have maximal rank.*

This statement makes sense because zero-dimensional schemes of given isomorphism class form an irreducible family. In Section 5 we prove the theorem, and we also give a precise description of the kind of schemes obtained as limits in the collision above. Once this is done, we get as a corollary the following.

Theorem 1.2. *Let (D, \mathbf{m}) be a union of singularity types of multiplicity two (whose weighted diagrams are of the types shown in Fig. 1). Then singularity schemes of type (D, \mathbf{m}) in general position have maximal rank in degrees $d \geq 13$.*

To give a complete proof of the result, Theorem 5.3, claimed above we need to deal with the (finitely many) cases that involve degrees 12 or less, which we do case by case with ad-hoc methods.

Most of the paper is devoted to the proof of Theorem 1.1, which is done—as said—by providing a sequence of specializations. The specializations are relatively easy to describe, and the main difficulty relies in computing the limit of an explicit one-parameter family of zero-dimensional schemes. To do this we rely in an algebraic lemma in the spirit of Alexander and Hirschowitz [3] or Évain [10].

2. An algebraic lemma

In this section, we state and prove a slightly generalized version of the “differential Horace lemma” of [3, Section 8]. The generalization, which is quite natural and more or less implicit in the works of Alexander–Hirschowitz [1–3], Évain [12,13], Mignon [19,20] and others, allows us to deal with more general families of zero-dimensional schemes, whose defining ideals are not “vertically graded,” as required in the cited papers.

Let R be an integral k -algebra, and consider $R_t = R \otimes k[[t]]$. Given $f_t \in R_t$, denote $f_0 \in R$ its image by the morphism $t \mapsto 0$. Similarly, for an ideal I_t in R_t , denote $I_0 = (I_t + (t))/(t) \subset R_t/(t) \cong R$.

Given an ideal $I_t \in R_t$, an element $y \in R$ and an integer $p \geq 1$, the p -trace and p -residual ideals of I_t with respect to y are defined as follows:

$$\begin{aligned}\mathrm{Tr}_p(I_t|y) &= \frac{((I_t + (y)) : t^{p-1})_0}{(y)} \subset R/(y), \\ \mathrm{Res}_p(I_t|y) &= ((I_t + (t^p)) : y)_0 \subset R.\end{aligned}$$

Note that there are inclusions $\mathrm{Tr}_1(I_t|y) \subset \mathrm{Tr}_2(I_t|y) \subset \cdots$, and $\mathrm{Res}_1(I_t|y) \supset \mathrm{Res}_2(I_t|y) \supset \cdots \supset I_0$. The ideals we are interested in have generically finite colength; we define $\mathrm{tr}_p(I_t|y) = \dim_k((R/(y))/\mathrm{Tr}_p(I_t|y))$ and $\mathrm{res}_p(I_t|y) = \dim_k(R/\mathrm{Res}_p(I_t|y))$.

As in [3], given any linear subspace $V \subset R$ and $y \in R$, let $\mathrm{Res}(V|y) = \{v \in R \mid vy \in V\}$. Since y is not a zero-divisor, we get a residual exact sequence

$$0 \rightarrow \mathrm{Res}(V|y) \xrightarrow{y} V \rightarrow V/V \cap (y) \rightarrow 0.$$

Proposition 2.1. *Let $V \subset R$ be a k -linear subspace, and $I_t \subset R_t$ an ideal such that R_t/I_t is flat over $k[[t]]$. Let $p \in \mathbb{Z}$ and $y \in R$ be given, with $p \geq 1$. Consider the following three canonical maps:*

$$\frac{V}{V \cap (y)} \xrightarrow{\varphi_p} \frac{R/(y)}{\mathrm{Tr}_p(I_t|y)}, \quad \mathrm{Res}(V|y) \xrightarrow{\check{\varphi}_p} \frac{R}{\mathrm{Res}_p(I_t|y)}, \quad V \otimes k[[t]] \xrightarrow{\varphi_t} R_t/I_t.$$

If φ_p is injective, then $(\mathrm{Ker} \varphi_t)_0 \subset y \mathrm{Ker} \check{\varphi}_p$.

Proof. Let $f_t \in \text{Ker } \varphi_t = V \otimes k[[t]] \cap I_t$. If $f_t \in (t^p, y)$, i.e., $f_t = g_t y + h_t t^p$ for some $g_t, h_t \in R_t$ then by the definitions $g_0 \in \text{Res}(V|y) \cap \text{Res}_p(I_t|y) = \text{Ker } \check{\varphi}_p$, and therefore $f_0 = y g_0 \in y \text{Ker } \check{\varphi}_p$, so it will be enough to prove that the injectivity of φ_p implies $f_t \in (t^p, y)$.

Write $f_t = \sum F_j t^j$, with $F_j \in R$. Denote \bar{F}_j the class of F_j in $R/(y)$; we want to see that $\bar{F}_0 = \dots = \bar{F}_{p-1} = 0$.

The inclusions $\text{Tr}_0(I_t|y) \subset \text{Tr}_1(I_t|y) \subset \dots$ together with the injectivity of φ_p tell us that, for every $j = 1, \dots, p$, the map

$$\varphi_j : \frac{V}{V \cap (y)} \rightarrow \frac{R/(y)}{\text{Tr}_j(I_t|y)}$$

is injective. As we have $f_t \in I_t$, it follows that

$$\bar{F}_0 \in \frac{I_t + (y, t)}{(y, t)} = \text{Tr}_1(I_t|y),$$

i.e., $\varphi_1(\bar{F}_0) = 0$, and therefore $\bar{F}_0 = 0$. Now we argue by iteration: let $1 \leq i < p$, and assume we have proved $\bar{F}_0 = \dots = \bar{F}_{i-1} = 0$. This means that $f_t \in (y, t^i)$, so $\sum_{j \geq i} F_j t^{j-i} \in (I_t + (y)) : t^i$, which implies $\bar{F}_i \in \text{Tr}_{i+1}(I_t|y)$, i.e., $\varphi_{i+1}(\bar{F}_i) = 0$, and therefore $\bar{F}_i = 0$. The proof is now complete. \square

3. Monomial ideals

In this paper, the ring R above will be the completion of the local ring at a given point of a smooth algebraic surface, and therefore isomorphic to a power series ring $R \cong k[[x, y]]$. In particular it is a regular local ring, and $R_t \cong k[[x, y, t]]$ is a regular local ring as well. Their maximal ideals are $\mathfrak{m} = (x, y)$ and $\mathfrak{m}_t = (x, y, t)$, respectively.

We shall be dealing with a restricted kind of ideals, of the form

$$I_E = (x^{e_1} f^{e_2})_{(e_1, e_2) \in E},$$

where $E \subset \mathbb{Z}_{\geq 0}^2$ is a *staircase*, that is, $E + \mathbb{Z}_{\geq 0}^2 \subset E$ (see [11]), and $f = x + y + t$. More generally, in some instances we shall consider ideals of the form

$$I_{(E, f, g)} = (g^{e_1} f^{e_2})_{(e_1, e_2) \in E},$$

where $f, g \in R$ are arbitrary. For convenience, we introduce some language to deal with the combinatorics of staircases.

Definition 3.1. If $E \subset \mathbb{Z}_{\geq 0}^2$ is a staircase, we say that the *length* of its i th stair is $\ell_E(i) = \inf\{e \mid (e, i) \in E\}$, and the *height* of its i th “slice” is $h_E(i) = \inf\{e \mid (i, e) \in E\}$. We shall use the first differences of ℓ and h as well: $\hat{\ell}_E(i) = \ell_E(i) - \ell_E(i+1)$, $\hat{h}_E(i) = h_E(i) - h_E(i+1)$.

When E is a staircase with finite complement, $h_E, \ell_E, \hat{\ell}_E$ and \hat{h}_E are functions $Z_{\geq 0} \rightarrow Z_{\geq 0}$, and each of them determines E uniquely.

Lemma 3.2. *Let E be a staircase, and let $\alpha, \beta \in \mathbb{Z}$ be such that $\hat{\ell}(i) \leq \alpha$ for all i with $\ell(i) \neq 0$ and $\hat{h}(i) \leq \beta$ for all i with $h(i) \neq 0$. Let $f, g, f', g' \in R$ be such that $(f, g) = (f', g')$, $f - f' \in (f, g)^\alpha$ and $g - g' \in (f, g)^\beta$. Then $I_{(E, f, g)} = I_{(E, f', g')}$.*

Proof. If E is empty then $I_{(E, f, g)} = (0) = I_{(E, f', g')}$ and there is nothing to prove; otherwise the hypotheses on the lengths of its stairs imply that E has finite complement.

We are going to prove that if $\hat{h}(i) \leq \beta$ for all i with $h(i) \neq 0$ and $g - g' \in (f, g)^\beta$ then $I_{(E, f, g')} \subset I_{(E, f, g)}$. Then it will follow that $g - g' \in (f, g')^\beta = (f, g)^\beta$ and hence $I_{(E, f, g)} \subset I_{(E, f, g')}$; thus $I_{(E, f, g')} = I_{(E, f, g)}$. By symmetry then it will follow that if $\hat{\ell}(i) \leq \alpha$ for all i with $\ell(i) \neq 0$ and $f - f' \in (f, g)^\alpha$ then $I_{(E, f', g)} = I_{(E, f, g)}$. Finally, $I_{(E, f, g)} = I_{(E, f', g')}$ as claimed.

Let us see that for all $(e_1, e_2) \in E$, $g^{e_1} f^{e_2} \in I_{(E, f, g)}$. We do it by induction on e_1 . If $e_1 = 0$ then there is nothing to prove; assume that $e_1 > 0$ and $g^{e'_1} f^{e'_2} \in I_{(E, f, g)}$ for all $(e'_1, e'_2) \in E$ with $e'_1 < e_1$.

Put $h = g' - g \in (f, g)^\beta$. Then

$$g^{e_1} f^{e_2} = \sum_{i=0}^{e_1} \binom{e_1}{i} g^i h^{e_1-i} f^{e_2}.$$

As $h^{e_1-i} \in (f, g)^{\beta(e_1-i)}$, we shall be done if $g^{i+a} f^{e_2+b} \in I_{(E, f, g)}$ for every $0 \leq i \leq e_1$ and every $a, b \geq 0$ with $a + b = \beta(e_1 - i)$ or, equivalently, if $(i + a, e_2 + b) \in E$ for every $0 \leq i \leq e_1$ and every $a, b \geq 0$ with $a + b = \beta(e_1 - i)$. But it follows from the hypothesis on β that $(a', b') \in E$ whenever $b' \geq e_2$ and $\beta a' + b' \geq \beta e_1 + e_2$, which is easy to check for $(a', b') = (i + a, e_2 + b)$. \square

The computation of quotient ideals of monomial ideals I_E as above leads, under suitable conditions, to new monomial ideals obtained by slicing off part of the staircase. This fact has already been exploited by Alexander–Hirschowitz and Évain, and we shall take advantage of it as well. So define $\sigma(E, p)$ as the staircase obtained from E by deleting the p th slice, i.e., the unique staircase whose height function has

$$h_{\sigma(E, p)}(i) = \begin{cases} h_E(i) & \text{if } i \leq p, \\ h_E(i + 1) & \text{if } i > p. \end{cases}$$

Proposition 3.3. *Let $E \subset \mathbb{Z}_{\geq 0}^2$ be a staircase, and $I_t = I_E$ as defined above. Assume that $\hat{\ell}_E(i) \geq 2$ for all $i < h_E(0) - 1$. Then*

- (1) R_t/I_t is flat over $k[[t]]$ and over $k[[y]]$,
- (2) $\text{tr}_p(I_E|y) = h_E(p - 1)$,
- (3) $\text{Res}_p(I_t|y) = (I_{\sigma(E, p)} + (t))/(t)$.

Proof. To prove the first claim, consider the automorphism ψ of R_t defined by $\psi(x) = x$, $\psi(y) = f$, $\psi(t) = t$. It is a $k[[t]]$ -automorphism (it leaves $k[[t]]$ fixed) so R_t/I_t is flat over $k[[t]]$ if and only if $R_t/\psi^{-1}(I_t)$ is. But $\psi^{-1}(I_t)$ is generated by monomials in x and y , so $R_t/\psi^{-1}(I_t) = R/\psi^{-1}(I_t)_0 \otimes k[[t]]$ is obviously flat over $k[[t]]$. The same argument, reversing the roles of y and t , proves the flatness over $k[[y]]$.

It has already been remarked and used (see [3, Section 8.1], [11]) and it is not hard to prove directly that for

$$J_E = ((y+t)^{e_1} x^{e_2})_{(e_1, e_2) \in E},$$

$\mathrm{tr}_p(J_E + (t^q)|y) = \mathrm{tr}_p(J_E|y) = h_E(p-1)$, and $\mathrm{Res}_p(J_t|y) = (J_{\sigma(E,p)} + (t))/(t)$.

Consider now the automorphism ψ of R_t defined by $\psi(x) = f$, $\psi(y) = y$, $\psi(t) = t$. Leaving t fixed, it induces an automorphism of $R \cong R_t/(t)$ which we also denote by ψ . It is not hard to see that the hypothesis $\hat{\ell}_E(i) \geq 1$ for all $i < h_E(0) - 1$ implies that $\hat{h}_E(i) \leq 1$ for all i , and therefore the previous lemma tells us that $I_E = ((y+t)^{e_1} f^{e_2})_{(e_1, e_2) \in E}$, so $\psi^{-1}(I_E) = J_E$, and therefore again $\mathrm{tr}_p(I_E|y) = \mathrm{tr}_p(J_E|y) = h_E(p-1)$, as desired.

Similarly, we have $\mathrm{Res}_p(I_t|y) = \psi(\mathrm{Res}_p(J_t|y)) = \psi(J_{\sigma(E,p)} + (t)/(t))$. As $\hat{\ell}_E(i) \geq 2$ for all $i < h_E(0) - 1$, it follows immediately that $\hat{\ell}_{\sigma(E,p)}(i) \geq 1$ for all $i < h_E(0) - 1$, and therefore $\psi(J_{\sigma(E,p)}) = ((y+t)^{e_1} f^{e_2})_{(e_1, e_2) \in \sigma(E,p)} = I_{\sigma(E,p)}$, finishing the proof. \square

4. Adjacencies in the Hilbert scheme

Consider now a point $O \in \mathbb{P}^2$, its blowing-up $\pi_O : S \rightarrow \mathbb{P}^2$, and a point O' in the first (infinitesimal) neighborhood of O (i.e., $O' \in S$ and $\pi_O(O') = O$). Let $\mathcal{O}_{O',S}$ be the local ring at O' and $y \in \mathcal{O}_{O',S}$ a local equation of the exceptional divisor $D = \pi_O^{-1}(O)$. To every $f, g \in \mathcal{O}_{O',S}$ and every staircase E we associate the ideal $I_{(E,f,g)}$ as defined in the previous section and for every integer m its (m -twisted) push-forward

$$J_{(E,f,g,m)} = (\pi_O)_*(y^m I_{(E,f,g)}) \subset \mathcal{O}_{O,\mathbb{P}^2}.$$

If f and g have no common components, and E has finite complement, then $J_{(E,f,g,m)}$ is \mathfrak{m} -primary (where \mathfrak{m} is the maximal ideal of $\mathcal{O}_{O,\mathbb{P}^2}$) and hence it defines a zero-dimensional subscheme $Z_{(E,O,O',f,g,m)}$ of \mathbb{P}^2 supported at O .

We define *2-curvilinear schemes* to be those schemes locally contained in the double of a curve; more precisely, a zero-dimensional scheme Z is called 2-curvilinear if it satisfies the following properties:

- (1) Z has embedding dimension at most 2; i.e., for every maximal ideal \mathfrak{m} of the Artinian ring \mathcal{O}_Z , $\dim_k \mathfrak{m}/\mathfrak{m}^2 \leq 2$, and
- (2) for every maximal ideal \mathfrak{m} of \mathcal{O}_Z , there exists $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ with $f^2 = 0$. Such an f is not unique, and we shall assume that it has been chosen of maximal contact with Z , i.e., that for every $g \in \mathfrak{m} \setminus \mathfrak{m}^2$ with $g^2 = 0$, $\dim_k (\mathcal{O}_Z/(f))_{\mathfrak{m}} \geq \dim_k (\mathcal{O}_Z/(g))_{\mathfrak{m}}$.

Moreover, to every such Z we attach invariants $N = \dim_k \mathcal{O}_Z$ (length) and ℓ (contact), which if Z is irreducible can be computed as $\ell = \dim_k(\mathcal{O}_Z/(f))$, where f is the chosen $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ with $f^2 = 0$ (for the unique maximal ideal). If Z has several components then its invariants are simply the sum of the invariants of each component.

Our interest in schemes $Z_{(E,O,O',f,g,m)}$ arises from the fact that some of them are specializations of (unions of) singularity schemes of multiplicity two. More precisely, they sit (inside $\text{Hilb } \mathbb{P}^2$) in the closure of the (irreducible) subscheme parameterizing 2-curvilinear schemes. In order to see this, we begin by showing that the $Z_{(E,O,O',f,g,m)}$ form a “nice” subset of $\text{Hilb } \mathbb{P}^2$. To simplify, assume that $g = x \in \mathcal{O}_{O',S}$ is transverse to D (i.e., x, y are a system of parameters of $\mathcal{O}_{O',S}$), and denote $s = \text{ord}(f|_D) = \dim_k((\mathcal{O}_{O',S}/(y, f)))$ the intersection multiplicity of $f = 0$ with the exceptional divisor. Introduce the notation

$$H_{m,E,s} = \left\{ Z_{(E,O,O',f,g,m)} \left| \begin{array}{l} O \in \mathbb{P}^2; \pi_O : S_O \rightarrow \mathbb{P}^2 \text{ is the blowing-up of } O; \\ O' \in D = \pi_O^{-1}(O); f, g \in \mathcal{O}_{O',S_O}; \\ (f, g) = \mathfrak{m}_{O',S_O}; \text{ord}(g|_D) = 1; \text{ord}(f|_D) = s. \end{array} \right. \right\}$$

Lemma 4.1. *Let $O \in \mathbb{P}^2$, $O' \in S$, $x, y, g \in \mathcal{O}_{O',S}$ be given as above. Put $s = \text{ord}(f|_D)$, let E be a staircase with finite complement, and let $m_0 = \min\{e_1 + se_2 \mid (e_1, e_2) \in E\}$. Then for every integer $m \geq m_0 - 1$,*

$$\dim_k \frac{\mathcal{O}_{\mathbb{P}^2, O}}{J_{(E,x,f,m)}} = \text{length}(m, E) := \binom{m+1}{2} + \#(\mathbb{Z}_{\geq 0} \setminus E).$$

Proof. Let $X \subset S$ be the zero-dimensional scheme defined by $I_{E,f,x}$ (so $X_{\text{red}} = O'$). It is clear that $\text{length } X = \#(\mathbb{Z}_{\geq 0} \setminus E)$ and, denoting by $D = \pi_O^{-1}(O)$ the exceptional divisor, $\text{length}(X \cap D) = m_0$. Thus the claim follows by [9, 2.14]. It is also possible to prove it along the lines of [7, 4.7.1]. \square

Lemma 4.2. *Let E be a staircase of height two and finite complement, and let $m_0 = \min\{e_1 + se_2 \mid (e_1, e_2) \in E\}$. For every integer $s \geq 1$ and $m \geq m_0 - 1$, the set*

$$H_{m,E,s} \subset \text{Hilb}^{\text{length}(m,E)} \mathbb{P}^2,$$

is an irreducible constructible subset for the Zariski topology.

Proof. Let $N = \text{length}(m, E)$. The claim will follow from the existence of a morphism $X \rightarrow \text{Hilb}^N \mathbb{P}^2$ where X is a smooth quasiprojective variety, whose image is $H_{m,E,s}$.

Assume that $\hat{\ell}_E(0) > 0$. Recall from [23] that the set of all clusters with given Enriques diagram has a natural structure of quasiprojective variety. Let $r = 1 + \max(\hat{\ell}_E(0), \hat{\ell}_E(1))$, and consider the diagram $D_s(r)$ shown in Fig. 2. Take $X = Cl(D_s(r))$ and define the map $Z : X \rightarrow \text{Hilb}^N \mathbb{P}^2$ as follows.

Given $K \in X$, let $O = p_0(K)$, $O' = p_1(K)$, and remark that O' is in the first neighborhood of O . Let $y \in \mathcal{O}_{O',S_O}$ be a local equation of the exceptional divisor of blowing up O , and choose a transverse germ $x = 0$ not going through $p_2(K)$. Choose $f \in \mathcal{O}_{O',S_O}$ to

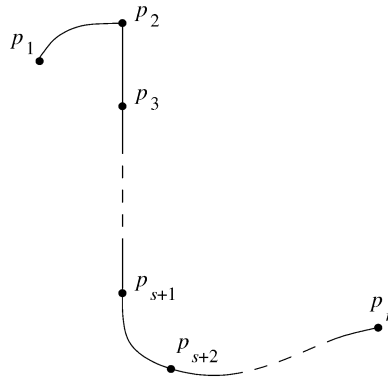


Fig. 2. Every point is proximate to the previous one; in addition, the s points after the origin are proximate to it (so p_3, \dots, p_{s+1} are satellite).

be a local equation of a germ of curve smooth at O' and going through all points of K . Then we set $Z(K) = Z_{(E, O, O', f, x, m)}$. Note that by the assumption that $\hat{\ell}_E(0) > 0$ we have $\hat{h}_E(i) \leq 1$ for all i and therefore by Lemma 3.2 $I_{(E, f, x)}$ does not depend on the choice of x , so neither does $Z(K)$. Similarly, the definition of r guarantees that $Z(K)$ does not depend on the choice of f .

It remains to be seen that the constructed map $Z: X \rightarrow \text{Hilb}^N \mathbb{P}^2$ is algebraic, and it is enough to do it locally.

Let $K_0 \in X$ be a closed point, and $p_1(K_0) \in \mathbb{P}^2$ the base point of the corresponding cluster. If (u, v) are affine coordinates in a neighborhood U_0 of $p_1(K_0) \in \mathbb{P}^2$, we may choose coordinates (u, v, x, y) in a neighborhood U_1 of $p_2(K_0)$ in the variety X_1 of all clusters of two points, in such a way that

$$\begin{aligned} U_1 &\xrightarrow{\psi_1} U_0, & U_1 &\xrightarrow{\pi_1} U_0, \\ (u, v, x, y) &\mapsto (u, v), & (u, v, x, y) &\mapsto (u + x, v + xy), \end{aligned}$$

are local expressions of the structure morphism and the relative blowing-up morphism of [23], i.e., π_1 restricted to the fiber of ψ_1 over $p \in U_0 \subset \mathbb{P}^2$ is (an affine chart of) the blowing up of p , and $y = 0$ is a local equation of the relative exceptional divisor, i.e., its restriction to each fiber of ψ_1 is a local equation of the corresponding exceptional divisor.

With these coordinates, there is a function $f = y + a_s x^s + \dots + a_{r-1} x^{r-1} \in \mathcal{O}(X_1)$ whose restriction to the fiber of ψ_1 over $p_1(K_0)$ vanishes at all points of K_0 [7], and therefore $Z(K_0) = Z_{(E, p_1(K_0), p_2(K_0), f, x, m)}$.

Then there are affine coordinates $(u, v, \bar{x}, x_s, \dots, x_{r-1})$ in a neighborhood U_{r-1} of K_0 in $Cl(D_s(r))$ such that the restriction of

$$\tilde{f} = y + (a_s + x_s)(\bar{x} - x)^s + \dots + (a_{r-1} + x_{r-1})(\bar{x} - x)^{r-1} \in \mathcal{O}(U_{r-1} \times_{U_0} U_1)$$

to the fiber over $p_1(K)$ is a local equation of a curve going through all the points of K (if $K \in U_{r-1}$ has coordinates $(u, v, \bar{x}, x_s, \dots, x_{r-1})$).

Consider now the ideals

$$\begin{aligned}\mathcal{I}_{(E, \tilde{f}, x)} &= (x^{e_1} \tilde{f}^{e_2})_{(e_1, e_2) \in E} \subset \mathcal{O}(U_{r-1} \times_{U_0} U_1), \\ \mathcal{J}_{(E, \tilde{f}, x, m)} &= (\text{id} \times \pi_1)_*(y^m \mathcal{I}_{(E, \tilde{f}, x)}) \subset \mathcal{O}(U_{r-1} \times U_0).\end{aligned}$$

Observe that (as in the previous lemma) for every $K \in U_{r-1}$, m_0 is the length of the intersection (in the corresponding fiber) of the zero-scheme defined by the restriction of $\mathcal{I}_{(E, \tilde{f}, x)}$ with the exceptional divisor, so by [9, 2.14] and [16, 7], $\mathcal{O}(U_{r-1} \times U_0)/\mathcal{J}_{(E, \tilde{f}, x, m)}$ is flat over $\mathcal{O}(U_{r-1})$ of relative length N , and it determines a morphism $U_{r-1} \rightarrow \text{Hilb}^N U_0 \subset \text{Hilb}^N \mathbb{P}^2$ which is set-theoretically equal to Z above.

It remains to deal with the case $\hat{\ell}_E(0) = 0$, in which the ideal $I_{(E, f, x)}$ does depend on the choice of x (in fact, on the class of x modulo the square of the maximal ideal of $p_2(K)$). This choice can be parameterized by the set of free points in the first neighborhood of p_2 . Thus one gets a map $X \rightarrow \text{Hilb}^N \mathbb{P}^2$ as before, with $X = Cl(D_s(r)) \times_{X_1} Cl(D_1(3))$, which can be shown to be a morphism in the same way. We leave the details to the interested reader. \square

Theorem 4.3. *Let E be a staircase of height two and s a positive integer satisfying $\hat{\ell}_E(0) \geq s + 2$ and $\ell_E(1) \geq s$. Define E_1 to be the unique staircase of height (at most) two with $\ell_{E_1}(0) = \ell_E(0) - s - 1$, $\ell_{E_1}(1) = \ell_E(1) - s$. If $\ell_E(1) \geq 2s + 1$, define furthermore E_2 to be the unique staircase of height (at most) two with $\ell_{E_2}(0) = \ell_E(0) - 2s - 2$, $\ell_{E_2}(1) = \ell_E(1) - 2s - 1$.*

If $\ell_E(1) \leq 2s - 2$ then $H_{2s+1, E_1, s+1} \subset \overline{H_{2s, E, s}}$, and if $\ell_E(1) \geq 2s - 1$ then $H_{2s+2, E_2, s+1} \subset \overline{H_{2s, E, s}}$.

Proof. Let $i = 1$ if $\ell_E(1) \leq 2s$ and $i = 2$ if $\ell_E(1) \geq 2s + 1$. Let $Z \in H_{m+i, E_i, s+1}$ be given by the ideal

$$J = J_{(E_i, f, x, m+i)} = (\pi_O)_*(y^{m+i} I_{(E_i, f, x)}) \subset \mathcal{O}_{O, \mathbb{P}^2},$$

where O is some point in \mathbb{P}^2 , O' is a point in the first neighborhood of O , $x, y, f \in \mathcal{O}_{O', S_O}$ are smooth germs, x, y are local parameters, $y = 0$ is a local equation of the exceptional divisor D , and $\text{ord}(f|_D) = s + 1$. Consider $f_t = f + tx^s \in \mathcal{O}_{O', S_O} \otimes k[t]$. For values of t in a neighborhood of 0, $J_t = (\pi_O)_*(y^m I_{(E, f, x)}) \subset \mathcal{O}_{O, \mathbb{P}^2}$ defines a zero-dimensional scheme Z_t in $H_{2s, E, s}$, so if we see that Z is the flat limit of Z_t , $t \rightarrow 0$, we shall be done. By 4.1, Z and Z_t have the same length, so it will be enough to show that $Z \supset \lim Z_t$ or, equivalently, that for every $g_t \in J_t$, $g_0 \in J$.

Denote by $W \subset \mathcal{O}_{O', S_O}$ the set of virtual transforms of equations of germs with (virtual multiplicity) at least m at O , i.e., $W = \pi^*(\mathfrak{m}^m)/y^m$. If we show that for every $g_t \in J_t \cap W \otimes k[t]$, $g_0 \in y^i I_{(E_i, f, x)}$, we shall be done.

Consider the (infinitely near) base points of the ideal I_t for each t [7, p. 254]. By hypothesis $\ell_E(1) \geq s$ and $\hat{\ell}_E(0) \geq s$, so looking at the Newton polygon of elements in I_t we see that there are at least s double base points on the germ $f_t = 0$; on the other hand, $\text{ord}(f_t|_D) \geq s$ so these base points do not depend on t and lie on the exceptional divisor.

Let $\tilde{\pi} : S \rightarrow S_O$ be the blowing up of the s base points, and let $P \in S$ be the point where the last exceptional divisor meets the strict transform \tilde{D} of D .

To simplify matters and to be able to use the results of Section 2, we pass to the completion, as we may. So let us denote $\mathcal{O} = \hat{\mathcal{O}}_{P,S}$ and let $\hat{x}, \hat{y} \in \mathcal{O}$ be local equations of the last exceptional divisor and of \tilde{D} , respectively. We require in addition that $\tilde{\pi}^* f = \hat{x}^s (\hat{x} + \hat{y}) \cdot u$, where u is a unit (this is not restrictive, since all ordinary singularities of multiplicity three are analytically equivalent). Then $\hat{x} + \hat{y} + t$ differs from \tilde{f}_t by a unit, where $\tilde{f}_t = (\tilde{\pi}^* f_t) / \hat{x}^s$. Let \hat{I}_t be the completion of the virtual transform of I_t with multiplicity two at the s base points; i.e., $\hat{I}_t = \tilde{I}_t \otimes_{\mathcal{O}_{P,S} \otimes k[[t]]} (\mathcal{O} \otimes k[[t]])$ where

$$\tilde{I}_t = \frac{\tilde{\pi}^*(I_t)}{\hat{x}^{2s}} \subset \mathcal{O}_{P,S} \otimes k[[t]].$$

With these notations, it is not hard to see that

$$\hat{I}_t = I_{(\hat{E}, \hat{x} + \hat{y} + t, \hat{x})},$$

where \hat{E} is obtained from E by shortening the stair lengths by s , i.e., $\hat{\ell}_{\hat{E}}(i) = \hat{\ell}_E(j) - s$, $j = 0, 1$.

Denote by $V \subset \mathcal{O}_{P,S} \subset \mathcal{O}$ the set of virtual transforms of elements of W with (virtual) multiplicity at least 2 at the s blown up points, i.e.,

$$V = \frac{\pi_*(\pi_*((\hat{x}^s \hat{y})^m \hat{x}^{2s}))}{(\hat{x}^s \hat{y})^m \hat{x}^{2s}} \otimes_{\mathcal{O}_{P,S}} \mathcal{O},$$

where $\pi = \pi_O \circ \tilde{\pi}$ is the composition of the blow ups.

Remark that, by the proximity equality [7, Theorem 3.5.3], every $g \in V$ not multiple of \hat{y} is the virtual transform at P of a germ at O which has multiplicity exactly $m = 2s$ and therefore does not vanish at P (i.e., $g \notin (x, y)$). So if $T \subset \mathcal{O}/(y)$ is a proper ideal of $\mathcal{O}/(y)$ then the canonical map

$$\frac{V}{V \cap (y)} \xrightarrow{\varphi_P} \frac{R/(y)}{T}$$

is injective.

Now let $g_t \in I_t \cap (W \otimes k[[t]])$, and let $\hat{g}_t = \tilde{\pi}^*(g_t) / \hat{x}^{2s} \in \hat{I}_t \cap (V \otimes k[[t]])$ be its virtual transform. We have that $\hat{\ell}_{\hat{E}}(0) \geq 2$, so we may apply Proposition 3.3 with $p = \ell_E(0) - 2s$, which gives $\text{tr}_p(\hat{I}_t | \hat{y}) = 1$ and then Proposition 2.1 shows that $\hat{g}_0 \in \hat{y} I_{(\sigma(\hat{E}, p), \hat{f}, \hat{x})}$. So g_0 is a multiple of y : $g_0 = yh$. Then $\hat{g}_0 = \tilde{\pi}^*(g_0) / \hat{x}^{2s} = \hat{y} \hat{x}^s \tilde{\pi}^*(h) / \hat{x}^{2s}$, with $\tilde{\pi}^*(h) / \hat{x}^s \in I_{(\sigma(\hat{E}, p), \hat{f}, \hat{x})}$. In other words, $h \in \tilde{\pi}_*(\hat{x}^s I_{(\sigma(\hat{E}, p), \hat{f}, \hat{x})}) = I_{(E_1, f, x)}$, so in the case $\ell_E(1) \leq 2s - 2$ we are done.

Assume now that $\ell_E(1) \geq 2s - 1$. In this case all elements $h \in I_{(E_1, f, x)}$ as above have multiplicity at least $m + 2$ along D . If y does not divide h , then $yh \in W$ tells us that h is the strict transform at O' of a germ at O which has multiplicity exactly $m + 1$, which

again contradicts the proximity equality. So y must divide h , and therefore $g_0 = y^2 h'$ with $h' \in (I_{(E_1, f, x)} : y) = I_{(E_2, f, x)}$ and we are done. \square

Theorem 4.3 is the main result on specialization inside the Hilbert scheme that we shall use. As said above, our interest in the schemes parameterized by the $H_{m, E, s}$ comes from the fact that they lie in the border of the subscheme of $\text{Hilb } \mathbb{P}^2$ parameterizing 2-curvilinear schemes. We now proceed to show this.

Lemma 4.4. *Let Z be a zero-dimensional scheme supported at a single point $O \subset \mathbb{P}^2$ and contained in a double curve $2C$, with C smooth at O . Let $N = \text{length } Z$, $\ell = \text{length } Z \cap C$, and let E be the staircase of height two with $\ell(0) = \ell$, $\ell(1) = N - \ell$. Let $y = 0$, $y \in \mathcal{O}_{O, \mathbb{P}^2}$ be a local equation for C , and let $x \in \mathcal{O}_{O, \mathbb{P}^2}$ be transverse, so that (x, y) is the maximal ideal of $\mathcal{O}_{O, \mathbb{P}^2}$. Then there exists a flat family of zero-dimensional schemes $Z_t \subset \mathbb{P}^2 \times \mathbb{A}^1$ such that $Z_1 = Z$, Z_t is isomorphic to Z for $t \neq 0$, and Z_0 is defined by the ideal $I_{(E, y, x)}$.*

Proof. Let $I \subset \mathcal{O}_{O, \mathbb{P}^2} \subset k[[x, y]]$ be the ideal defining Z . As I is (x, y) -primary, we may safely pass to the completion $\hat{\mathcal{O}}_{O, \mathbb{P}^2} \cong k[[x, y]]$. It is immediate that $I_{(E, y, x)}$ is the initial ideal of I with respect to the *negative lexicographical ordering* with $1 > x > y$ [14, Example 1.2.8]. The desired family is then given by flat deformation to the initial ideal (see, for instance, [14, Theorem 7.5.1]). \square

Lemma 4.5. *Let Z be a zero-dimensional scheme contained in a double curve $2C$, with C smooth at Z_{red} . Let $N = \text{length } Z$, $\ell = \text{length } Z \cap C$, and let E be the staircase of height two with $\ell(0) = \ell$, $\ell(1) = N - \ell$. Let $O \in C$ be an arbitrary point, let $y = 0$, $y \in \mathcal{O}_{O, \mathbb{P}^2}$ be a local equation for C , and let $x \in \mathcal{O}_{O, \mathbb{P}^2}$ be transverse, so that (x, y) is the maximal ideal of $\mathcal{O}_{O, \mathbb{P}^2}$. Then there exists a flat family of zero-dimensional schemes $Z_t \subset \mathbb{P}^2 \times \mathbb{A}^1$ such that $Z_1 = Z$, Z_t is isomorphic to Z for $t \neq 0$, and Z_0 is defined by the ideal $I_{(E, y, x)}$.*

Proof. Use Lemma 4.4 and Évain's “collision de front” [12]. \square

Corollary 4.6. *Let Z_0 be a 2-curvilinear zero-dimensional scheme, with invariants N and ℓ , and let $H(Z_0) \subset \text{Hilb } \mathbb{P}^2$ be the set of all zero-dimensional subschemes of the plane isomorphic to Z_0 . For every positive integer k , let E_k be the staircase of height two and $\ell_{E_k}(0) = \ell - k(k+1)$, $\ell_{E_k}(1) = N - \ell - k^2$. Then $H(Z_0) \subset \text{Hilb } \mathbb{P}^2$ is constructible in the Zariski topology, and for every $k \geq 1$ satisfying $2\ell - N > 2k$, $\ell \geq k(k+1)$, $N - \ell \geq k^2$, one has $\overline{H(Z_0)} \supset H_{2k, E_k, k}$.*

Proof. That $H(Z_0)$ is constructible is a general fact that does not use 2-curvilinearity of Z_0 . For the claimed incidences, observe first that 4.5 immediately gives $\overline{H(Z_0)} \supset H_{2, E_1, 1}$, so for $k = 1$ we are done. Now proceed by recurrence on k , observing that for $k > 1$, the hypotheses imply $\hat{\ell}_{E_{k-1}}(0) \geq k+1$ and $\ell_{E_{k-1}}(1) \geq 2k-3$, so Theorem 4.3 tells us that $H_{2k, E_k, k} \subset \overline{H_{2k-1, E_{k-1}, k-1}}$. \square

5. Hilbert function of 2-curvilinear schemes

In order to prove our main theorem it is now enough to identify the cases in which the sequence of specializations of the previous section has led us to a maximal rank scheme type. To begin with, let us recall a known class of maximal rank schemes:

Lemma 5.1. *Let E be a staircase of height two and m a positive integer satisfying $\ell_E(0) > m$ and $2\ell_E(1) \leq m$. Then for every s such that $m \geq \min\{e_1 + se_2 \mid (e_1, e_2) \in E\}$, general elements of $H_{m,E,s}$ have maximal rank.*

Proof. It is not hard to see that the schemes parameterized by $H_{m,E,s}$ are cluster schemes (they are defined by complete ideals). Then it is enough to compute their cluster of base points to see that the claim is equivalent to [22, Lemma 4.4]. \square

Proof of Theorem 1.1. Note first that if $N - \ell \leq 1$ then either $N = \ell$ and elements of $H_{m,E,s}$ are curvilinear, in which case the result is well known, or due to Lemma 4.5 it is enough to prove the maximal rank for general elements of $H_{2,E,\ell-2}$, with E of height 1, which is again well known. See [22, Lemma 4.3] for a proof that covers both cases over a field of arbitrary characteristic. Other proofs for the curvilinear case can be found in the literature; two elegant options over \mathbb{C} are [8], which works in arbitrary dimension, or the use of Briançon's specializations of [6].

So assume that $N - \ell \geq 2$ and let k be the maximal integer such that $N - \ell \geq k^2$. The hypothesis of the theorem tells us that $\ell \geq (N - \ell) + 1 + 3\sqrt{N - \ell} \geq k^2 + k$ and $2\ell - N \geq 1 + 3\sqrt{N - \ell} \geq 2k$, so we may apply Corollary 4.6 and it will be enough to prove that general members of $H_{2k,E_k,k}$, with E_k as in 4.6, have maximal rank. We distinguish two cases. Assume first that $N - \ell \leq k(k + 1)$. Then it follows that $\ell_{E_k}(1) \leq k$ and $\ell_{E_k}(0) \geq (N - \ell) - k^2 + 1 + 3\sqrt{N - \ell} - k \geq 2k + 1$, and Lemma 5.1 finishes the proof.

Assume now that $N - \ell \geq k(k + 1) + 1$. Then $\ell_{E_k}(1) \geq k + 1$ and $\hat{\ell}_{E_k}(0) \geq k + 3$ so we may apply Theorem 4.3 and obtain $\bar{H}_{2k,E_k,k} \supset H_{2k+1,E'_k,k+1}$, where E'_k has height two and $\ell_{E'_k}(0) = \ell - (k + 1)^2 - 1$, $\ell_{E'_k}(1) = N - \ell - k(k + 1)$. So it is enough to prove that general members of $H_{2k+1,E'_k,k+1}$ have maximal rank. But $\ell_{E'_k}(0) = \ell - (k + 1)^2 - 1 \geq (N - \ell) - (k + 1)^2 + 3\sqrt{N - \ell} > 2k$ and $\ell_{E'_k}(1) = N - \ell - k(k + 1) < k + 1$ (because the choice of k gives $N - \ell < (k + 1)^2$) and again we finish using Lemma 5.1. \square

Proof of Theorem 1.2. We want to apply Theorem 1.1. Clearly a scheme $Z_{(K,m)}$ is 2-curvilinear (as explained in the introduction) but we need some bounds on the length N and maximal contact ℓ to hold.

If (D, \mathbf{m}) is the diagram of an A_{2k-1} singularity, $k \geq 1$, and $K \in Cl(D)$ has $p_1(K) = O \in \mathbb{P}^2$, let $(x, y) \in \mathcal{O}_{O,\mathbb{P}^2}$ be a system of parameters such that $y = 0$ is the equation of a smooth germ of curve going through all the points of K . Then the ideal of $Z_{(K,m)}$ is (y^2, yx^k, x^{2k}) , and its invariants are $N = 3k$, $\ell = 2k$. Similarly, if (D, \mathbf{m}) is the diagram of an A_{2k-2} singularity, $k \geq 2$, $K \in Cl(D)$ has $p_1(K) = O \in \mathbb{P}^2$, and $(x, y) \in \mathcal{O}_{O,\mathbb{P}^2}$ are a system of parameters such that $y = 0$ is the equation of a smooth germ of curve going through all the points of K but the last, then the ideal of $Z_{(K,m)}$ is (y^2, yx^k, x^{2k-1}) , and

its invariants are $N = 3k - 1$, $\ell = 2k - 1$. All in all, the invariants of a union of singularity schemes always satisfy $5\ell \geq 3N$.

Therefore a scheme as in the claim, being the union of such schemes, still satisfies $\ell \geq (3/5)N$. So in order to prove that $N \leq 2\ell - 1 - 3\sqrt{N - \ell}$ it would be enough that $N \leq (6/5)N - 1 - 3\sqrt{(2/5)N}$. This inequality is always satisfied if $N \geq 100$ and then the claim follows from Theorem 1.1. Now assume that $N < 100$; we need to prove the independence of the conditions imposed by a general $Z_{(K, \mathbf{m})}$ to curves of degree 13 (and therefore of higher degree as well). Consider the scheme X union of $Z_{(K, \mathbf{m})}$ and $100 - N$ reduced points in general position. X is still a 2-curvilinear scheme, and has invariants $N' = 100$, $\ell' = \ell + 100 - N \geq (3/5)N'$ so it is of maximal rank as before. The linear system of all curves of degree 13 has dimension $104 > 100$, so X does impose independent conditions to curves of degree 13, and hence $Z_{(K, \mathbf{m})} \subset X$ does too. \square

Finally, let us deal with the low degree cases. First of all, there are a number of (well known) exceptions.

Remark 5.2. The following schemes are *not* of maximal rank in the mentioned degrees:

- (1) Two ordinary double points in general position, in degree 2.
- (2) Two ordinary cusp schemes in general position, in degree 3.
- (3) A union of nodes or tacnodes A_{2k_i-1} in general position with $\sum k_i = 5$, in degree 4 (a particular case of which is 5 double points mentioned in the introduction).

Theorem 5.3. Let (D, \mathbf{m}) be a union of weighted diagrams of types A_k (shown in Fig. 1), not among the exceptions 5.2. Then for K general in $Cl(D)$, $Z_{(K, \mathbf{m})}$ has maximal rank.

Proof. Because of Theorem 1.2, we only need to show that the given schemes have maximal rank in degrees less than 13. Let N and ℓ be the invariants associated to schemes $Z_{(K, \mathbf{m})}$ for $K \in Cl(D)$. We can assume that $5\ell \geq 3N$. Let d be the maximal integer such that $d(d+1)/2 < N$, and let $N' = d(d+1)/2$, $N'' = (d+1)(d+2)/2$, $\ell' = \min\{\ell, d(d+1)/2\}$, $\ell'' = \ell + d(d+1)/2 - N$ then it is very easy to see that a scheme $Z \in H_{2, E(N, \ell), 1}$ contains a $Z' \in \text{Hilb}^{N'} \mathbb{P}^2$ and is contained in a $Z'' \in \text{Hilb}^{N''} \mathbb{P}^2$, and both are unions of singularity schemes of multiplicity two and (possibly) simple points (in particular, they are still 2-curvilinear and their invariants satisfy $5\ell \geq 3N$). Moreover, if Z is in general position then we may assume that Z' and Z'' are in general position too. Thus, reasoning as in [18] or [22], it is enough to prove that every union of singularity schemes of multiplicity two and simple points, in general position, of length $N = (d+1)(d+2)/2$ for $d \leq 12$, has maximal rank.

The cases with $N \leq 2\ell - 1 - 3\sqrt{N - \ell}$ have already been solved. In particular for $N = (12+1)(12+2)/2$ all cases with $\ell \geq 55$ are done, and these are the only ones with

$5\ell \geq 3N$, so we can assume $d \leq 11$. We consider the cases with $d \leq 4$ to be well known. Putting everything together the cases we are left with have the following invariants:

d	N	ℓ
11	78	47
10	66	40
9	55	33, 34
8	45	27, 28
7	36	22, 23
6	28	17, 18
5	21	13, 14

The cases $N = 55$, $\ell = 33$ and $N = 45$, $\ell = 27$ can only be realized by singularity schemes consisting of 11 and 9 ordinary cusps, respectively. Thus they were solved by Barkats [5].

The case $N = 78$, $\ell = 47$ leads, after the sequence of specializations used in the proof of Theorem 1.1, to schemes in $H_{10,E,5}$, where E is the height two staircase with $\ell_E(0) = 17$, $\ell_E(1) = 6$. These are cluster schemes (defined by complete ideals) which have maximal rank by [22, Proposition 4.5], so we are done. The same method works for the cases $N = 55$, $\ell = 34$, $N = 36$, $\ell = 23$ and $N = 21$, $\ell = 14$.

The case $N = 66$, $\ell = 40$ can only be realized by singularity schemes consisting of 10 ordinary cusps plus some other singularity schemes. So a scheme Z in this class can be written as $Y \cup Z'$ where Y is a cusp scheme and Z' is a 2-curvilinear singularity scheme with invariants $N' = 61$, $\ell' = 37$, both in general position. After the sequence of specializations used in the proof of Theorem 1.1, Z' degenerates into a scheme Z'_0 in $H_{8,E,4}$, where E is the height two staircase with $\ell_E(0) = 17$, $\ell_E(1) = 8$, so Z degenerates into $Y \cup Z'_0$. Specialize now the position of Y so that its tangent line meets Z'_0 , which forces curves of degree 10 containing $Y \cup Z'_0$ to contain the line as well. It is not hard to check that in fact these curves consist of the line tangent to Y counted twice plus curves of degree 8 through a residual scheme of maximal rank and degree 45 (Z'_0 is in fact a cluster scheme so residuals are easily computed) so there are no such curves. By semicontinuity then there are no curves of degree 10 containing Z either, and we are done. The same argument solves cases $N = 45$, $\ell = 28$ and $N = 36$, $\ell = 22$ (which can only be realized by schemes one of whose components is an ordinary cusp scheme). Not all cases with $N = 28$, $\ell = 18$ have one ordinary cusp as a component, but those which do have one are also solved.

For the remaining cases we only state the specializations that lead to the solution, leaving to the reader the actual computations.

If $N = 28$, $\ell = 18$ and no component is an ordinary cusp, let $3 \leq k \leq 5$ be minimal such that one component is the scheme of a singularity of type A_{2k-2} (analytically equivalent to $y^2 - x^{k+1}$). If $k = 3$, specialize this component to a scheme in $H_{3,E_1,2}$, where E_1 has height 1 and length 2, and the rest of the components (as in Theorem 1.1) to a scheme in $H_{4,E_2,2}$, where E_2 has height 2 and stairs of lengths 13 and 6. The line joining the two components is a fixed part of the system under consideration and allows to conclude. If $k = 4$, specialize the rest of the components (as in Theorem 1.1) to a scheme in $H_{4,E,2}$, where E has height 2 and lengths 5 and 2. The line joining the two components is a fixed part of the system under consideration and allows to conclude. If $k = 5$, then there are exactly two components both of type A_6 . Specialize one of them (as in Theorem 1.1) to

a scheme in $H_{4,E,2}$, where E has height 2 and lengths 3 and 2, and specialize further so that it is supported on the unique conic of maximal contact with the other component. This conic and the line tangent to the scheme of $H_{4,E,2}$ are fixed parts of the system under consideration and allow to conclude.

If $N = 28$, $\ell = 17$, there must be 4 ordinary cusp schemes involved. Specialize two of them to be supported at the tangent line of a third; this line is a fixed component and allows to conclude.

Finally, if $N = 21$, $\ell = 13$ there must be at least one ordinary cusp. If there are two, then the rest of the components can be specialized to a scheme of a singularity of type A_6 with contact 6 with a line, which allows to conclude. If there is only one, then one other component must be the scheme of a singularity of type A_4 ; specialize so that the cusp is supported at the tangent line to the A_4 . \square

References

- [1] J. Alexander, A. Hirschowitz, La méthode d'Horace éclatée: application à l'interpolation en degré quatre, *Invent. Math.* 107 (1992) 586–602.
- [2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, *J. Algebraic Geom.* 4 (1995) 201–222.
- [3] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.* 140 (2000) 303–325.
- [4] V.I. Arnold, Local normal forms of functions, *Invent. Math.* 35 (1976) 87–109.
- [5] D. Barkats, Variétés des courbes planes à noeuds et à cusps, in: P.E. Newstead (Ed.), *Algebraic Geometry, Proceedings of the Europroj Conferences in Catania and Barcelona*, Dekker, 1998, pp. 25–36.
- [6] J. Briançon, Description de $\text{Hilb}^n \mathbb{C}\{x, y\}$, *Invent. Math.* 41 (1977) 45–89.
- [7] E. Casas-Alvero, *Singularities of Plane Curves*, London Math. Soc. Lecture Note Ser., vol. 276, Cambridge Univ. Press, Cambridge, 2000.
- [8] C. Ciliberto, R. Miranda, Interpolation on curvilinear schemes, *J. Algebra* 203 (1998) 677–678.
- [9] P. Deligne, Intersections sur les surfaces régulières, in: SGA 7 II, *Groupes de Monodromie En Géométrie Algébrique*, in: *Lecture Notes in Math.*, vol. 340, Springer-Verlag, 1973, pp. 1–38.
- [10] L. Évain, Collisions de trois gros points sur une surface algébrique, PhD thesis, Université de Nice, 1997.
- [11] L. Évain, Dimension des systèmes linéaires: une approche différentielle et combinatoire, preprint, math-ph/9705018, 1997.
- [12] L. Évain, Une minoration du degré de courbes planes à singularités imposées, *Bull. Soc. Math. France* 126 (1998) 525–543.
- [13] L. Évain, La fonction de Hilbert de la Réunion de 4^h gros points génériques de \mathbb{P}^2 de même multiplicité, *J. Algebraic Geom.* (1999) 787–796.
- [14] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, Springer-Verlag, Berlin, 2002, With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh, and UNIX).
- [15] G.M. Greuel, C. Lossen, E. Shustin, Plane curves of minimal degree with prescribed singularities, *Invent. Math.* 133 (1998) 539–580.
- [16] A. Grothendieck, Éléments de géométrie algébrique, iii: Étude cohomologique des faisceaux cohérents, seconde partie, *Publ. Math.*, vol. 17, Inst. Hautes Études Sci., 1963.
- [17] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., vol. 52, Springer-Verlag, 1977.
- [18] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables, *Manuscripta Math.* 50 (1985) 337–388.
- [19] T. Mignon, Courbes lisses sur les surfaces rationnelles génériques: un lemme d'Horace différentiel, *Ann. Inst. Fourier (Grenoble)* 50 (6) (2000) 1709–1744.

- [20] T. Mignon, An asymptotic existence theorem for plane curves with prescribed singularities, *J. Algebraic Geom.* 10 (2) (2001) 281–297.
- [21] F. Palatini, Sulla rappresentazione delle forme ternarie mediante somme di potenze di forme lineari, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Ser. V* 12 (3) (1903) 378–384.
- [22] J. Roé, On the conditions imposed by tacnodes and cusps, *Trans. Amer. Math. Soc.* 353 (12) (2001) 4925–4948.
- [23] J. Roé, Varieties of clusters and Enriques diagrams, *Math. Proc. Cambridge Philos. Soc.* 137 (1) (2004) 69–94.
- [24] E. Shustin, Analytic order of singular and critical points, *Trans. Amer. Math. Soc.* 356 (3) (2004) 953–985, MR MR1984463.